An Numerical Approach to Robust Partial Quadratic Eigenvalue Assignment in Second-Order Control Systems

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Figure: Wobbling of the Millennium Footbridge Over the River Thames

Instability of Vibration Structures

- Resonance: The structure is excited by external forces whose frequencies are close to its natural frequencies. The vibrations are amplified and the system becomes unstable.

- Vibration control of structures (e.g. bridges, highways, and aircrafts) are essential to achieve optimal design with desirable performance.

- Traditional Method: Passive damping treatment. Reliable, robust, and without significantly altering the structural mass and stiffness but unadjustable.

- More Popular Method: Active Vibration Control. Adjustable and significant damping.

Applications of Active Vibration Control

- Large flexible space structure control;
- Earthquake engineering;
- Control of flexible multibody systems;
- Controller design for damped gyroscopic systems;
- Vibration in structural dynamics.

Main Goals of Active Vibration Control

- Determine a state feedback controller to reduce vibrations (e.g. eliminate the resonant frequencies);
- Keep crucial system properties unchanged.

Modelling:

$$M\ddot{\boldsymbol{x}}(t) + D\dot{\boldsymbol{x}}(t) + K\boldsymbol{x}(t) = B\boldsymbol{u}(t).$$

• The dynamical characteristics of a structure are governed by the natural frequencies and mode shapes, i.e., the eigenvalues and eigenvectors of the quadratic eigenvalue problem:

$$P(\lambda)\boldsymbol{x} := (\lambda^2 M + \lambda C + K)\boldsymbol{x} = 0.$$

• The control force

$$\boldsymbol{u}(t) = F_1^T\dot{\boldsymbol{x}} + F_2^T\boldsymbol{x}(t)$$

leads to the close-loop pencil

$$P_c(\lambda) := \lambda^2 M + \lambda(D - BF_1^T) + (K - BF_2^T) = \mathbf{0}.$$

Partial Quadratic Eigenvalue Assignment Problem:

• Find the feedback matrices F_1 and F_2 such that the few unwanted eigenvalues

$$\lambda_1, \ldots, \lambda_p \, (p \ll 2n)$$

are reassigned to the desired ones:

$$\mu_1, \ldots, \mu_p.$$

• The other $2n - p$ eigenvalues and eigenvectors of the open-loop pencil $P(\lambda)$ are preserved (*no spill-over phenomenon*).

Remarks

• Retaining the "acceptable" (e.g. no-resonant) eigenvalues and eigenvectors can ensure that there is no spurious modes will be introduced into the frequency range of interests;

• One may transform the second-order model to the first-order control system

$$\dot{\boldsymbol{z}}(t) = \hat{A}\boldsymbol{z}(t) + \hat{B}\boldsymbol{u}(t),$$

where

$$\hat{A} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix}, \quad \boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{x}(t) \\ \dot{\boldsymbol{x}}(t) \end{bmatrix}.$$

Remarks

• The PQEAP is converted the pole assignment problem for the first-order control systems where many numerical methods are available, see Chu'07;

• Computational Drawbacks:

- Increase in dimensionality from n to $2n$;
- Inversion of a possibly ill-conditioned mass M;
- loss of exploitable structures such as the symmetry, definiteness, sparsity and bandedness.

Therefore, the PQEAP is substantially different from the first-order pole assignment problem.

Minimization Norm and Robust PQEAP

From a practical point of view, it is desirable to determine feedback matrices F_1 and F_2 such that.

- The feedback norm is minimized. small feedback gains lead to smaller control signals, and thus to less energy consumption. Also, small gains reduce noise amplification.
- The sensitivity of the close-loop eigenvalues is minimized. This is *robust* PQEAP.

Challenges:

- The problem must be considered in the quadratic setting without the transformation to the first order state space forms.
- Only the few reassigned open-loop eigenvalues and the associated eigenvectors are available.
- No spill-over phenomenon should be guaranteed.
- No model reduction is allowed.
- The gradient formulas of the objective function should be in terms of the few known eigenvalues and eigenvectors only.

Notations

- $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$: The eigenvalues to be reassigned;
- $X_1 = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p]$: The matrix of the eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_p$;
- $\Lambda_1' = \mathrm{diag}(\mu_1, \ldots, \mu_p)$: The eigenvalues to assign;
- $\Lambda_2 = \mathrm{diag}(\lambda_{p+1}, \ldots, \lambda_{2n})$: The $2n - p$ open-loop eigenvalues (unavailable);
- $X_2 = [\boldsymbol{x}_{p+1}, \ldots, \boldsymbol{x}_{2n}]$: The matrix of the eigenvectors corresponding to the eigenvalues $\lambda_{p+1}, \ldots, \lambda_{2n}$ (unavailable);
- Y_1: Close-loop eigenvectors corresponding to the eigenvalues $\mu_1, \ldots, \mu_p$.

Assumptions

- M, D, K real symmetric with $M > 0$.
- Partial Controllability of the model $(P(\lambda), B)$

$$\mathrm{rank}(\lambda_i^2 M + \lambda_i D + K, B) = n \text{ for } i = 1, \ldots, p.$$

- Sets of eigenvalues are closed under complex conjugation.
- $0 \notin \{\lambda_1, \ldots, \lambda_p\}$.

Parametric Expressions for the feedback Matrices

(Theorem:Brahma and Datta'07) The feedback matrices F_1 and F_2 are determined by

$$\begin{cases} F_1 = MX_1\Lambda_1\Phi^T \\ F_2 = -KX_1\Phi^T \end{cases}$$

where Φ satisfies

$$\Phi Z = \Gamma \; (\Gamma \text{ arbitrary})$$

with Z being the unique solution of the Sylvester equation:

$$\Lambda_1 Z - Z\Lambda_1' = -\Lambda_1 X_1^T B\Gamma.$$

Feedback Norm Minimization Problem

Let

$$S := [F_2^T, F_1^T]$$

$$\min \quad \Pi := \frac{1}{2}\|S\|_F^2 = \frac{1}{2}(\|F_1\|_F^2 + \|F_2\|_F^2)$$

- **Theorem for Gradient Formula (Brahma and Datta'07):**
  - Let $C := [-X_1^T K, \Lambda_1 X_1^T M]$. Then

$$S = \Gamma Z^{-1} C$$

  - Let U be the solution to the Sylvester equation

$$\Lambda_1' U - U\Lambda_1 = -Z^{-1} C S^H \Phi.$$

Then,

$$\nabla_\Gamma(\Pi) = 1/2[Z^{-1} C S^H - U\Lambda_1 X_1^T B]^T.$$

Robust PQEAP

The close-loop eigenvectors

$$Y = \begin{bmatrix} Y_1 & X_2 \\ Y_1\Lambda_1' & X_2\Lambda_2 \end{bmatrix}$$

where $Y_1 = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_p]$ with $\boldsymbol{y}_k$ satisfying $(\mu_k^2 M + \mu_k D + K)\boldsymbol{y}_k = B\gamma_k$.

$$\min \quad J := \|(I - Y^H Y)^2\|_F^2$$

- **Theorem for Gradient Formula (Brahma and Datta'07):**
  - $J = \|(I - Y^H Y)^2\|_F^2 = \|Z_1\|_F^2 + \|Z_2\|_F^2 := J_1 + J_2$, where

    $$Z_1 = I_p - Y_1^H Y_1 - \bar{\Lambda}_1' Y_1^H Y_1 \bar{\Lambda}_1, \quad Z_2 = I_{2n-p} - X_2^H X_2 - \bar{\Lambda}_2 X_2^H X_2 \bar{\Lambda}_2.$$

  - $$\nabla_\Gamma(J) = \nabla_\Gamma(J_1) \quad (\nabla_\Gamma(J_2) = 0),$$

    where $\nabla_\Gamma(J_1)$ is given in terms of Y_1, X_1, and Λ_1 only.

Parametric Solutions to PQEAP

Attn: The assumption that $0 \notin \{\lambda_1, \ldots, \lambda_p\}$ is canceled.

- The feedback matrices F_1 and F_2 are given by

$$\begin{cases} F_1 &= MX_1\Phi^T \\ F_2 &= (MX_1\Lambda_1 + DX_1)\Phi^T \end{cases}$$

where Φ satisfies

$$\Phi Z = \Gamma \; (\Gamma \text{ arbitrary})$$

with Z being the unique solution of the Sylvester equation:

$$\Lambda_1 Z - Z\Lambda_1' = -X_1^T B\Gamma.$$

Robust PQEAP

The close-loop eigenvectors

$$Y = \begin{bmatrix} Y_1 & X_2 \\ Y_1 \Lambda_1' & X_2 \Lambda_2 \end{bmatrix}$$

where $Y_1 = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_p]$ with $\boldsymbol{y}_k$ satisfying $(\mu_k^2 M + \mu_k D + K)\boldsymbol{y}_k = B\gamma_k$.

$$\min \quad J := \frac{1}{2}(1-\alpha)(\|F_1\|_F^2 + \|F_2\|_F^2) + \frac{1}{2}\alpha(\|Y\|_F^2 + \|Y^{-1}\|_F^2),$$

where $0 \leq \alpha \leq 1$.

Remarks

- Reduction of J:

$$\begin{aligned} J &= \underline{\tfrac{1}{2}\alpha(\|Y_1\|_F^2 + \|Y_1\Lambda_1'\|_F^2) + \tfrac{1}{2}\alpha\|Y^{-1}\|_F^2} \\ &+ \underline{\tfrac{1}{2}(1-\alpha)(\|F_1\|_F^2 + \|F_2\|_F^2)} \\ &+ \underline{\tfrac{1}{2}\alpha(\|X_2\|_F^2 + \|X_2\Lambda_2\|_F^2)} \\ &:= J_1 + J_2 + J_3. \end{aligned}$$

- Challenges:
  - Could we get an explicit expression for Y^{-1}?
  - How to find the gradient formula for the above cost function without knowing Λ_2 and X_2?

- If the formula is found, then an optimization technique such as Broyden-Fletcher-Goldfrab-Shanon (BFGS) method can be employed.

Expression of Y^{-1}

- **Theorem:** Let

$$\begin{cases} C & := & [\Lambda_1 X_1^T M + X_1^T D, X_1^T M], \\ \tilde{Y}_1 & := & [Y_1^T, \Lambda_1' Y_1^T]^T, \\ \tilde{X}_2 & := & [X_2^T, \Lambda_2 X_2^T]^T. \end{cases}$$

Then

$$Y^{-1} = \begin{bmatrix} Z^{-1}C \\ \tilde{X}_2^+(I - \tilde{Y}_1 Z^{-1} C) \end{bmatrix} := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}.$$

Comments

It follows that

$$\|W_2\|_F \leq \|\tilde{X}_2^+\|_2 \|(I - \tilde{Y}_1 W_1)\|_F = 1/\sigma_{\min}(\tilde{X}_2)\, \|I - \tilde{Y}_1 W_1\|_F.$$

By the assumption that $\tilde{X}_2$ has linearly independent columns, one may expect that $\sigma_{\min}(\tilde{X}_2)$ is not too small. Suppose that β is *a-priori* estimate of $1/\sigma^2_{\min}(\tilde{X}_2)$. Therefore, we shall minimize

$$\begin{aligned} J &= \underline{\tfrac{1}{2}\alpha(\|Y_1\|_F^2 + \|Y_1\Lambda_1'\|_F^2 + \|W_1\|_F^2 + \beta\|I - \tilde{Y}_1 W_1\|_F^2)} \\ &+ \underline{\tfrac{1}{2}(1-\alpha)(\|F_1\|_F^2 + \|F_2\|_F^2)} \\ &+ \underline{\tfrac{1}{2}\alpha(\|X_2\|_F^2 + \|X_2\Lambda_2\|_F^2)} \\ &:= J_1 + J_2 + J_3 \quad \text{(for simplicity)}. \end{aligned}$$

Gradient Formula

• **Theorem:** Let U, U_1, U_2, Υ, V, V_1, and V_2, respectively, satisfy the following equations

$$\begin{cases} MU\Lambda_1'^2 + DU\Lambda_1' + KU = [(I + \Lambda_1'\bar{\Lambda}_1')Y_1^H]^T, \\ MU_1\Lambda_1'^2 + DU_1\Lambda_1' + KU_1 = [W_1 W_{11}^H]^T, \\ MU_2\Lambda_1'^2 + DU_2\Lambda_1' + KU_2 = [\Lambda_1' W_1 W_{21}^H]^T \end{cases}$$

and

$$\begin{cases} \Lambda_1'\Upsilon - \Upsilon\Lambda_1 = -Z^{-1}CS^H\Phi \\ \Lambda_1' V - V\Lambda_1 = -W_1 W_1^H Z^{-1}, \\ \Lambda_1' V_1 - V_1\Lambda_1 = -W_1 W_{11}^H Y_1 Z^{-1}, \\ \Lambda_1' V_2 - V_2\Lambda_1 = -W_1 W_{21}^H Y_1 \Lambda_1' Z^{-1}, \end{cases}$$

Gradient Formula

where

$$W_{11} = E_1 - Y_1 W_1 \quad \text{and} \quad W_{21} = E_2 - Y_1 \Lambda_1' W_1$$

with $E_1 := [I_n, 0] \in \mathbb{R}^{n\times 2n}$ and $E_2 := [0, I_n] \in \mathbb{R}^{n\times 2n}$. Then,

$$\nabla_\Gamma J = \alpha \nabla_\Gamma J_1 + (1-\alpha)\nabla_\Gamma J_2,$$

where

$$\begin{aligned}
\nabla_\Gamma J_1 &= \tfrac{1}{2}[U^T B + V X_1^T B]^T \\
&- \tfrac{1}{2}\beta[(U_1 + U_2)^T B + (V_1 + V_2) X_1^T B]^T, \\
\nabla_\Gamma J_2 &= \tfrac{1}{2}[Z^{-1} C S^H - \Upsilon X_1^T B]^T.
\end{aligned}$$

Numerical Results

Problem 1:

- $M = \begin{bmatrix} 1.0000 & 0.0167 & -0.1428 & 0.1971 \\ 0.0167 & 1.0000 & -0.0524 & 0.5457 \\ -0.1428 & -0.0524 & 1.0000 & -0.2326 \\ 0.1971 & 0.5457 & -0.2326 & 1.0000 \end{bmatrix}$, $D = \begin{bmatrix} 1.0000 & -0.0527 & -0.2816 & -0.3331 \\ -0.0527 & 1.0000 & 0.3098 & -0.3367 \\ -0.2816 & 0.3098 & 1.0000 & 0.3715 \\ -0.3331 & -0.3367 & 0.3715 & 1.0000 \end{bmatrix}$

$K = \begin{bmatrix} 0.7330 & 0.4380 & -0.0860 & 0.6015 \\ 0.4380 & 0.3108 & 0.1600 & 0.1961 \\ -0.0860 & 0.1600 & 0.9198 & -0.7733 \\ 0.6015 & 0.1961 & -0.7733 & 1.0364 \end{bmatrix}$, $B = \begin{bmatrix} 0.1987 & 0.0153 \\ 0.6038 & 0.7468 \\ 0.2722 & 0.4451 \\ 0.1988 & 0.9318 \end{bmatrix}$

- The open-loop eigenvalues are:

$\{-3.4209, -0.1943 \pm 1.0642i, 0.0000, -0.5474 \pm 0.8820i, -1.0777, 0.0000\}$. The eigenvalues $\{0.0000, 0.0000\}$ were reassigned to $\{-0.3 \pm 1.5i\}$, the other eigenvalues were retained.

Numerical Results

Problem 2 (Nichols & Kautsky'01):

- $$M = 10I_3, \quad D = 0, \quad K = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}.$$

- The open-loop eigenvalues are: $\{\pm 3.6039i, \pm 2.4940i, \pm 0.8901i\}$. The first two eigenvalues $\{\pm 3.6039i\}$ were reassigned to $\{-1, -2\}$, the other eigenvalues were preserved.

Numerical Results

Problem 3 (Tisseur & Meerbergen'01):

- $$M = I_n, D = \tau \text{tridiag}(-1, 3, -1), K = \kappa \text{tridiag}(-1, 3, -1), B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$
- $n = 50$, $\tau = 3$, and $\kappa = 5$.
- The first six open-loop complex eigenvalues $\{-3.3306 \pm 0.0947i, -3.1628 \pm 0.7344i, -3.0000 \pm 1.0000i\}$ were reassigned to $\{-3.5 \pm 0.0947i, -3.5 \pm 0.7344i, -3.5 \pm 1.0000i\}$. The other 94 eigenpairs are kept unchanged.

Table: Numerical results for Problems 1–3 ("$e5$" means "$\times 10^5$")

	Mini-N	$\beta = 1$		$\beta = 10^2$		$\beta = 10^4$	
	$\alpha = 0$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 0.5$
$\|\|F_1\|\|_F$	20.09	12.88	12.38	21.78	21.12	18.96	21.77
$\|\|F_2\|\|_F$	23.40	12.19	12.73	51.05	49.00	40.92	51.04
$\kappa_2(Y)$	6.8e12	22.03	24.17	162.81	158.34	569.64	162.49^{P1}
$\|\|F_1\|\|_F$	19.36	26.58	21.31	26.87	30.11	26.88	26.93
$\|\|F_2\|\|_F$	70.89	99.68	72.23	99.52	95.67	99.52	99.47
$\kappa_2(Y)$	4.86e3	10.19	177.05	10.22	11.41	10.22	10.22^{P2}
$\|\|F_1\|\|_F$	5.24	18.23	23.41	273.42	584.51	28.94	28.94
$\|\|F_2\|\|_F$	18.22	74.04	92.41	$1.0 \cdot 10^3$	$2.3 \cdot 10^3$	122.13	122.13
$\kappa_2(Y)$	2.2e9	2.9e5	4.2e5	1.0e7	1.8e7	7.6e5	7.6$e5^{P3}$

Table: Numerical results for Problems 2 and 3 ($\mathtt{RE.} = 100 \times (\mathtt{IV.} - \mathtt{FV.})/\mathtt{IV.}$)

	Brahma-Datta's Robust Mehtod			Our method with $\alpha = 1, \beta = 10^2$		
	$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$	$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$
IV.	74.98	130.71	6.98×10^3	74.98	130.71	6.98×10^3
FV.	59.44	161.54	232.99	26.87	99.52	10.22
RE.	20.72	-23.58	96.66	64.16	23.86	99.85^{P2}
IV.	5.1067	17.83	8.37×10^3	5.1067	17.83	8.37×10^3
FV.	6.6371	21.84	3.28×10^3	5.1281	17.80	1.57×10^3
RE.	-29.97	-22.46	60.79	-0.42	0.20	81.30^{P3}

Thank You!